Ordinary Differential Equation (ODE)

INTRODUCTION: Ordinary Differential Equations play an important role in different branches of science and technology. In the practical field of application problems are expressed as differential equations and the solution to these differential equations are of much importance. In Here we shall discuss the fundamental concepts of ordinary differential equations followed by which deal with the various analytical methods to solve different forms of differential equations.

If x is the independent variable and y is a dependent variable then the equation involving x ,y and one or more of the following

$$\frac{dy}{dx}, \frac{d^2y}{dx^2}, \ldots, \frac{d^ny}{dx^n}, \ldots.$$

is called an ordinary differential equation.

Observation: The word ordinary states the fact that there is only one independent variable in the equation.

ORDER AND DEGREE OF ORDINARY DIFFEHENIAL EQUATIONS: The order of an ordinary differential equation is the order of the highest ordered derivative involved in the equation and the degree of an ordinary differential equation is the power of the highest ordered derivative after making the equation rational and integrable as far as derivatives are concerned.

Example1: The order and degree of the differential equation

$\frac{d^4y}{dx^4} + 7\frac{dy}{dx} + 3y = e^x$ is 4 and 1 respectively.

Example: Find a differential equation from the following relation

$$y = \frac{A}{x} + B.$$

Solution: Here we observed that here of constant are A and B we have to eliminate the constants.

Now, $y = \frac{A}{x} + B,$ ….(1)

Differentiating (1) with respect to x we get

$$\frac{dy}{dx} = -\frac{A}{x^2} \qquad \ldots(2)$$

Differentiating (2) with respect to x again we get

$$\frac{d^2y}{dx^2} = +\frac{2A}{x^3} \qquad \ldots(3)$$

Eliminating A from (2) and (3) we have

$$\frac{d^2y}{dx^2} = \frac{2}{x^3}\left(-x^2\frac{dy}{dx}\right).$$

Or, $\frac{d^2y}{dx^2} = -\frac{2}{x}\frac{dy}{dx}$,

Or, $\frac{d^2y}{dx^2} + \frac{2}{x}\frac{dy}{dx} = 0.$

Therefore the required differential equation is of second order, and which is of the reduced form

$\frac{d^2y}{dx^2} + \frac{2}{x}\frac{dy}{dx} = 0.$

Example2: Suppose a curve is defined by the condition that the sum of x and y intercepts of its tangents is always equal to m. Express this by a differential equation.

Solution The equation of the tangent at any point (x, y) on the curve is given by

$$Y - y = \frac{dy}{dx}(X - x)$$

Or, $\frac{X}{\frac{y - x\frac{dy}{dx}}{-\frac{dy}{dx}}} = \frac{Y}{y - x\frac{dy}{dx}} = 1$

By the given condition we have

$$\frac{y - x\frac{dy}{dx}}{-\frac{dy}{dx}} + y - x\frac{dy}{dx} = m$$

Or, $y - x\frac{dy}{dx} + x - y\frac{dx}{dy} = m$

Or, $x\left(\frac{dy}{dx}\right)^2 - (x + y - m)\frac{dy}{dx} + y = 0.$

This is the required first order but second degree differential equation.

<u>Ordinary differential equation of first order and first degree</u>

Any differential equation of first order and first degree is of the form

$$f(x, y) = \frac{dy}{dx}.$$

Let $f(x, y) = -\frac{M(x,y)}{N(x,y)}$,

Then the standard form of differential equation of first order is

$$M(x, y)dx + N(x, y)dy = 0.$$

The following equations represent first order, first degree:

1. $\frac{dy}{dx} = x + y.$
2. $(e^x \sin y + e^{-y})dx + (e^x \cos y - xe^{-y})dy = 0$

Theorem: I f $\frac{\partial M}{\partial y}$ and $\frac{\partial N}{\partial x}$ are continuous then the necessary and sufficient condition that the differential equation $M(x, y)dx + N(x, y)dy = 0$ to be exact is $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$.

Method of solution

Step 1: Calculate $\int M(x, y)dx$ treating y as constant.

Stem 2: Calculate $\int N(x, y)dy$ for those terms of N which do not contain x.

Stem 3: solution is

$\int M(x, y)dx$ treating y as constant+$\int N(x, y)dy$ for those terms of N which do not contain x = constant.

Example: Show that the differential equation

$(e^x \sin y + e^{-y})dx + (e^x \cos y + xe^{-y})dy = 0$

is exact differential equation and find the general solution.

Solution: Here $M(x, y) = e^x \sin y + e^{-y}$ and $N(x, y) = e^x \cos y + xe^{-y}$.

Now $\frac{\partial M}{\partial y} = e^x \cos y - e^y$ and $\frac{\partial N}{\partial x} = e^x \cos y - e^{-y}$.

Therefore

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ .The given equation is exact.

And its solution is $e^x \sin y + xe^{-y} = c$, where c is arbitrary constant.

<u>Ordinary Differential equation of Higher Order and first Degree</u>

The important and method of solution of first order differential equations have been discussed in previous articles. Differential applications of engineering and science encounter linear differential equations of higher order with constant coefficients and coefficients are functions of x.

Example: Solve the equation, $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = 0.$

Solution: Let $y = e^{mx}$ be a trial solution of $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = 0.$

Then the auxiliary equation

$m^2 - 4m + 4 = 0$. Hence $m = 2, 2$.

Hence the solution of the given equation, $y = (c_1 + c_2x)e^{2x}$.

Example: Solve $(D^2 - 4)y = \cos^2 x$. Where $D = \frac{d}{dx}$.

Solution: Let $y = e^{mx}$ be a trivial solution then the auxiliary equation is $m^2 - 4 = 0$,

Hence $m = \pm 2$.

Therefore complementary function is

$y_{CF} = C_1e^{-2x} + C_2e^{2x}$

Where C_1 and C_2 are constant.

Now the particular integral is

$$y_{PI} = \frac{1}{D^2-4} cos^2 x$$

$$= \frac{1}{D^2-4}\frac{1+\cos 2x}{2}$$

$$= \frac{1}{2}\frac{1}{D^2-4}1 + \frac{1}{2}\frac{1}{D^2-4}\cos 2x$$

$$= -\frac{1}{8}\left(1 - \frac{D^2}{4}\right)^{-1}.1 + \frac{1}{2}\frac{1}{(-2^2-4)}\cos 2x$$

$$= -\frac{1}{8}\left(1 + \frac{D^2}{4}\right).1 - \frac{1}{16}\cos 2x$$

$$= -\frac{1}{8} - \frac{1}{16}\cos 2x\ .$$

Therefore the general solution is

$$y = y_{CF} + y_{PI} = C_1e^{-2x} + C_2e^{2x} - \frac{1}{8} - \frac{1}{16}\cos 2x$$

Supplementary questions

1. Solve the following differential equation
   (a) $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = e^{2x} + x^3 + \cos 2x$.
   [Answer: $y = (c_1 + c_2x)e^{2x} + \frac{1}{2}x^2e^{2x} + \frac{1}{8}(2x^3 + 6x^2 + 9x + 6) - \frac{1}{8}\sin 2x$.]
   (b) $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = e^{-2x}\sin 2x$

[Answer: $y = (c_1 + c_2 x)e^x - \frac{1}{10}e^{-2x}(\cos 2x + 2\sin 2x)$.]

Example: Solve by variation of parameter,

$\frac{d^2y}{dx^2} + y = \sec^3 x \tan x$.

Solution: The differential equation can be written as

$(D^2 + 1)y = \sec^3 x \tan x$, where $D \equiv \frac{d}{dx}$.

Here the complementary function $y_{CF} = c_1 \cos x + c_2 \sin x$.

Let the particular integral be $y_{CF} = u \cos x + v \sin x$, where u and v are function of x.

Now $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos x & sinx \\ -\sin x & \cos x \end{vmatrix} = 1 \neq 0$.

Now $u = -\int \frac{y_2 X}{W} dx = -\int \sin x \, sec^3 \, x \, tan \, x \, dx = \frac{tan^3 x}{3}$.

And $v = \int \frac{y_1 X}{W} dx = \int \cos x \, sec^3 \, x \tan x \, dx = \frac{tan^2 \, x}{2}$.

Therefore the solution is $y = c_1 \cos x + c_2 \sin x + \frac{tan^3 x}{3} \cos x + \frac{tan^2 \, x}{2} \sin x$.

Graph Theory

Introduction to graphs:

Graph theory is one of the oldest subjects with lots of applications in applied mathematics and engineering. The great Swiss mathematician Leonhard Euler introduced the basic ideas of the subject in the eighteenth century. After that several numbers of research articles and books have been published in this field.

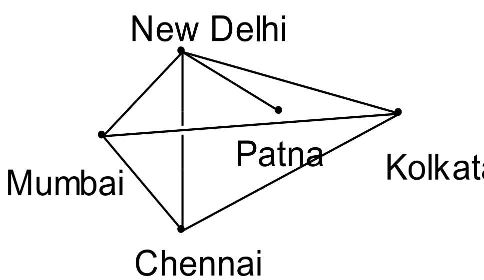
Graphs are used in many diverse fields, including Computer Science, Operations Research, Chemistry, Electrical Engineering, Linguistics and Economics.

In this chapter we begin our study with an introduction to several basic concepts in the theory of graphs and examples. A few results involving these concepts will also be established.

Graphs and their representations:

To represent situations involving some objects and their relationships by drawing a diagram of points with segments joining those points that are related. Let us consider some specific examples of this idea.

Example 1. Consider for a moment an airline route in which dots represent the cities and two dots are join if there is a direct flight between the corresponding cities. Such an airline map shown in the Fig.-6.1.



Definition. A graph (or a undirected graph) G is a pair (V_G, E_G) where V_G is the non empty finite set, called the set of vertices (or nodes) and E_G is a finite set (may be empty) whose elements are called the edges (or arcs) such that each edge e is identified with an unordered pair {u, v}(or simply uv) of vertices, i.e. $E_G = \{\{u, v\}: u, v \in V, u \neq v\}$ or . $E_G = \{uv: u, v \in V, u \neq v\}$.

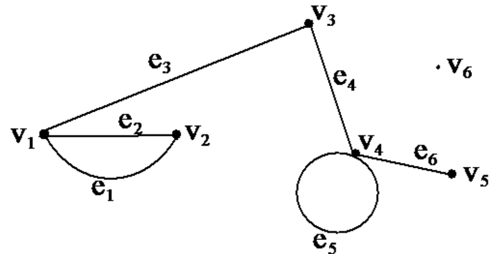
For a graph the number $|V_G|$ of the vertices is called the order of G and $|E_G|$ is the size of G. The vertices u, v associated with an edge 'e' is called end vertices of e. When a vertex v is an end vertex of some edge e, then v is said to be incident on e and e is said to be incidence with v.

Definition: A graph G = (V_G, E_G) is trivial, if it has only one vertex, i.e. $|V_G| = 1$; otherwise G is nontrivial.

Graph Terminology:

Before proceeding, we need following terminology about graphs.

1. **Adjacent vertices:** Two vertices are said to be adjacent if they are the end vertices of an edge. In the graph G in Fig.-1 $e_2 = v_1v_2$ is incidence with the vertices v_1 and v_2. So v_1 and v_2 are adjacent vertices. But, since no edge is incidence with v_2 and v_3 they are not adjacent.



Undirected Graph G

2. **Parallel Edges:** If some distinct edges are incidences with same pair of vertices then the edges are called parallel edges. In the graph G in Fig.-1 e_1 and e_2 are parallel edges incident with the vertex v_1 and v_2.

3. **Loop:** An edge incident on a single vertex is called a loop, i.e. if end vertices of an edge are same then that edge is called a loop. In the graph G in Fig.-1 e_5 is a self-loop.

4. **Isolated vertex:** A vertex, which is not the end vertex of any edge, called isolated vertex. In other wards isolated vertex is of degree zero. In the graph G in Fig.-1 v_6 is an isolated vertex.

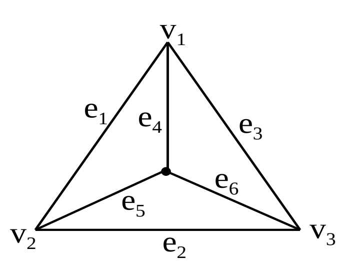
5. **Pendant vertex:** A vertex of degree one is called a pendant vertex. In the graph G in Fig.-1 v_5 is a pendant vertex.

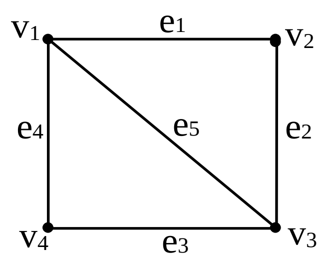
6. **Degree of a vertex:** The degree of a vertex in an undirected graph is the number of edges incident with it, except that a loop at a vertex contributes twice to the degree of that vertex. The degree of the vertex v is denoted by deg(v) or d(v).
   The number $\delta(G) = \min\{d(v) \mid v \in V_G\}$ is the minimum degree of G, the number $\Delta(G) = \max\{d(v) \mid v \in V_G\}$ is the maximum degree of G and $d_A(G) = \frac{1}{|V_G|}\sum_{v \in V_G} d(v)$ is the average degree of G.

Corollary 1. $\delta(G) \le \frac{2|E_G|}{|V_G|} \le \Delta(G)$.

Types of graphs: Graphs are classified according to the existence of multiple edges and loops.

1. **Simple graph:** A graph that has neither a loop nor parallel edges is called simple graph. The following figures are the examples of simple graph.





Multiple graph: The graph in which the same two vertices are joined by more than one edge are called multiple graphs or multigraphs.

The graphs, which have edges from a vertex to itself (i.e. self-loop) are not allowed in multigraphs. Instead of Pseudographs are used. Pseudographs are more general than multigraphs since they may contain loops and multiple edges.

Table – 1.

Type	Multiple edges allowed?	Loops allowed?
Simple Graph	No	No
Multigraph	Yes	No
Pseudograph	Yes	Yes

2. **Weighted graph:** The graph G is called a weighted graph if each edge e of G is assigned a non-negative number w(e) called the weight or length.

3. **Finite or infinite graph:** A graph is said to be finite if it has finite number of vertices and a finite number of edges. Observe that a graph with finite number of vertices must automatically have a finite number of edges and so must be finite; otherwise it is an infinite graph.

Theorem 1: If G is a graph with m edges and n vertices then $\sum_{i=1}^{n} d(v_i) = 2m$.

Proof: Let e be an edge with end vertices v_i and v_j then e contribute 1 to the degree of v_i and 1 to the degree of v_j. If e is a self-loop on v then e contributes 2 to the degree of v. Thus each edge contributes 2 to the total number of degrees of the vertices.

Hence, $\sum_{i=1}^{n} d(v_i) = 2m$.

Note: The above theorem sometimes called the **Handshaking theorem**, because of the analogy between an edge having two end points and a handshake involving two hands.

Theorem 2: The number of vertices of odd degree in a graph is always even.

Proof: Let V_1 and V_2 be the set of vertices of even degree and the set of vertices of odd degree respectively in a graph G = (V_G, E_G). Then by theorem -1

$$2m = \sum_{v \in V} d(v) = \sum_{v \in V_1} d(v) + \sum_{v \in V_2} d(v).$$

Since d(v) is even for each $v \in V_1$, the first term in the right-hand side is even. Furthermore, the sum of the terms of the right –hand side is even since 2m is even. Hence the second term is also even. Since all the terms in this sum are odd, there must be an even number of such terms. Thus there is even number of vertices of odd degree.

Problem 1: Show that the maximum degree of any vertex in a simple graph with n vertices is (n-1).

Since the graph is simple, no self-loop and parallel edges are present in it. So in maximum case in a simple graph one vertex can be connected with the remaining vertices.
So, in a simple graph with n vertices, a vertex can be connected with maximum (n-1) vertices and as the degree of a vertex is the number of edges incident on it, the maximum degree of any vertex in a simple graph with n vertices is (n-1).

Problem 2: Show that the maximum number of edges in a simple graph with n vertices is $\frac{n(n-1)}{2}$.

Solution: By Handshaking theorem $\sum_{i=1}^{n} d(v_i) = 2m$. Where m is the number of edges with n vertices in the graph G. $\Rightarrow d(v_1) + d(v_2) + \ldots\ldots + d(v_n) = 2m \ldots\ldots\ldots\ldots(1)$
since we know that the maximum degree of each vertices in the graph G can be (n-1).
Therefore equation (1) reduces to (for maximum case) (n-1) + (n-1) + …+ (n-1) = 2m

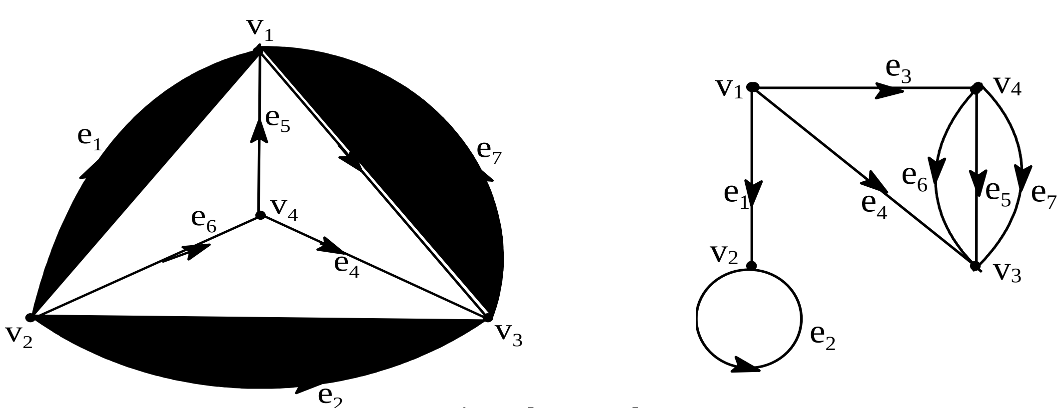
$\Rightarrow$ $n(n-1) = 2m \Rightarrow m = \frac{n(n-1)}{2}$.

$\Rightarrow$ Hence the maximum number of edges in any simple graph is $\frac{n(n-1)}{2}$.

Problem 3: Is it possible to draw a simple graph with 4 vertices and 7 edges?
Solution: The graph does not exit, since in a simple graph with n vertices the maximum number of edge with be $\frac{n(n-1)}{2}$. Here $\frac{4(4-1)}{2}$ =6 < 7.

Some Important Graphs:

1. **Directed graphs or Digraphs:** When a direction is associated with the edges of a graph. Formally, a digraph or oriented graph G(D) is an ordered pair (V_G(D), E_G(D)) where V_G(D) is a nonempty finite set of elements known as vertices and E_G(D) is a family of ordered pairs (not necessarily distinct) of elements known as directed edges or arcs.



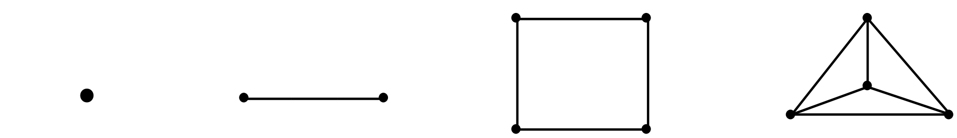
Digraphs G and H

The graph G in Fig -3 two edges joining v_1 and v_3 having opposite directions and without loops, hence this is a simple digraph. On the other hand in H, v_3 and v_4 are joined by three edges with same direction with a loop e_2, hence this is a multiple digraph.

In-degree and out-degree: In a graph with directed edges the in-degree of a vertex v, denoted by $d^{-}(v)$ or $\vec{d}(v)$ is the number of edges with v as their terminal vertex. The out degree of v, denoted by $d^{+}(v)$ or $\overleftarrow{d}(v)$, is the number of edges with v as their initial vertex. (Note that a loop at a vertex contributes 1 to both in degree and out degree of this vertex). The graph H in Fig– 3 in-degree and out-degree of different vertices are $d^{-}(v_1) = 0$, $d^{-}(v_2) = 2$, $d^{-}(v_3) = 4$, $d^{-}(v_4) = 1$, $d^{+}(v_1) = 3$, $d^{+}(v_2) = 1$, $d^{+}(v_3) = 0$, $d^{+}(v_4) = 3$.

Observation 1: In a digraph $G = (V_G, E_G)$, $\sum_{i=1}^{n} d^{-}(v_i) = \sum_{i=1}^{n} d^{+}(v_i) = |E_G|$.

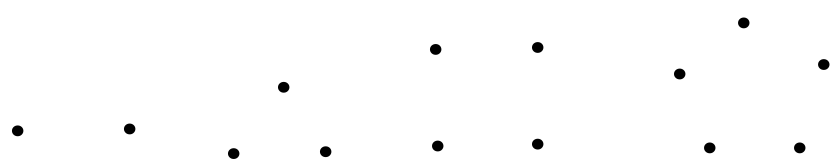
2. **Regular Graph:** Simple graph in which all vertices are of equal degree is called regular graph. A regular graph is also called n-regular if every vertex has degree n.



Regular graph of degree 0 | Regular graph of degree 1 | Regular graph of degree 2 | Regular graph of degree 3

Regular graphs

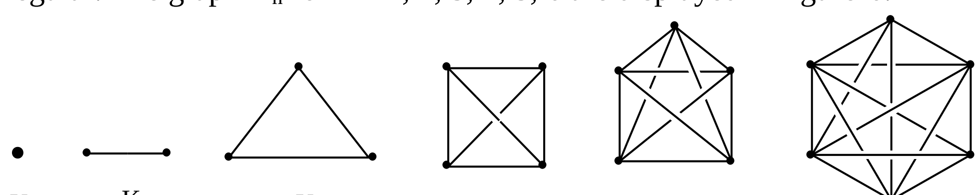
3. Null graph: A graph G is said to be null graph if edge set E of the graph is to be empty set. N_n denotes null graph with n vertices. The null graph N_2 N_3, N_4 and N_5 are displayed in Fig-5.



N_2 N_3 N_4 N_5

The null graphs N_2, N_3, N_4 and N_5

4. Complete graph: The complete graph on n vertices, denoted by K_n, is the simple graph that contains exactly one edge between each pair of distinct vertices. That is a complete graph K_n is (n – 1)- regular. The graph K_n for n = 1, 2, 3, 4, 5, 6 are displayed in figure-6.

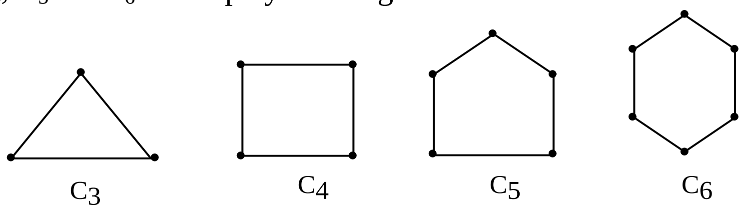


K_1 K_2 K_3 K_4 K_5 K_6

The Complete graphs K_1, K_2, K_3, K_4, K_5 and K_6

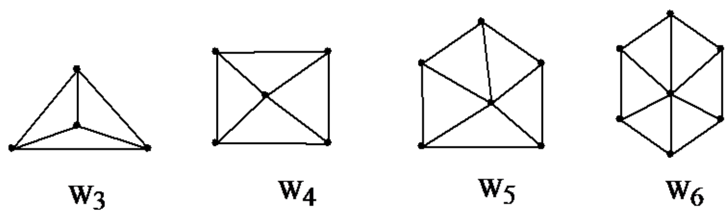
Note: The edges are sometimes may intersect at a point that does not represent a vertex, for example in K_4, diagonal edges have no common vertices.

5. Cycle: A graph on n($\geq$ 3) vertices, denoted by C_n, is said to be a cycle if the edges are v_1v_2, v_2v_3, v_3v_4, …., $v_{n-1}v_n$.
The cycles C_3, C_4, C_5 and C_6 are displayed in Fig.-7.



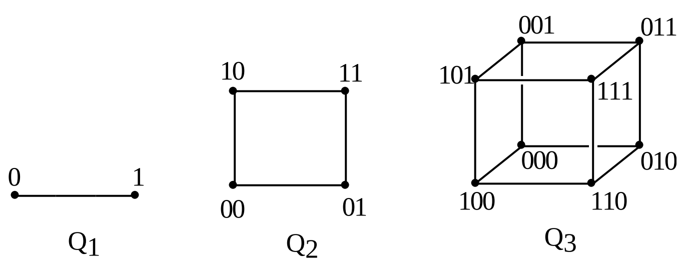
The cycles C_3, C_4, C_5 and C_6

6. Wheel: We obtain the wheel W_n when we add an additional vertex to the cycle C_n, (n $\geq$ 3) and connect this new vertex to each of the n vertices in C_n, by the new edges.
The wheels W_3, W_4, W_5 and W_6 are displayed in Fig.-8.



The wheel W_3, W_4, W_5 and W_6

7. n-Cubes: The n-cube, denoted by Q_n is the graph whose vertices representing the 2^n bit string (or order n- tuples of 0 and 1) of length n. Two vertices are adjacent if and only if they differ in exactly one bit position (one co-ordinate). The graph Q_1, Q_2 and Q_3 are displayed in Fig-9.



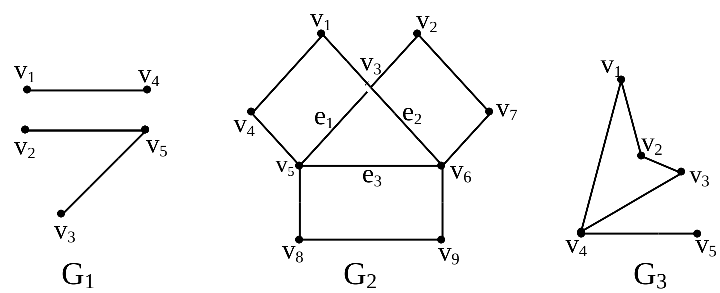
The n cube Q_n for n =1, 2, 3.

The order of Q_n is $|V_G| = 2^n$, the number of binary string of length n. Also since Q_n is n-regular, by Handshaking lemma $|E_G| = n\,2^{n-1}$.

8. Bipartite Graph: Some times a graph has the property that its vertex set can be displayed into two disjoint sub-sets such that each edge connects a vertex in one of this subset to a vertex in other sub-set. For example, consider the graph representing marriages between people in a state, where the vertices represent the persons and the marriages are represented by the edges. In this graph, each edge connects a vertex in the subset of vertices representing males and a vertex in the subset of vertices representing formulas.

Definition: A simple graph $G(V_G, E_G)$ is called a bipartite graph if the vertex set can be partitioned into two disjoint non-empty sets V_1 and V_2 such that each edge in E is incident on one vertex in V_1 and one vertex in V_2 (so that no edge in E connects either two vertices in V_1 or two vertices in V_2).

Example: States, which of the following graphs are bipartite? If the graph is bipartite specify the disjoint vertex sets.



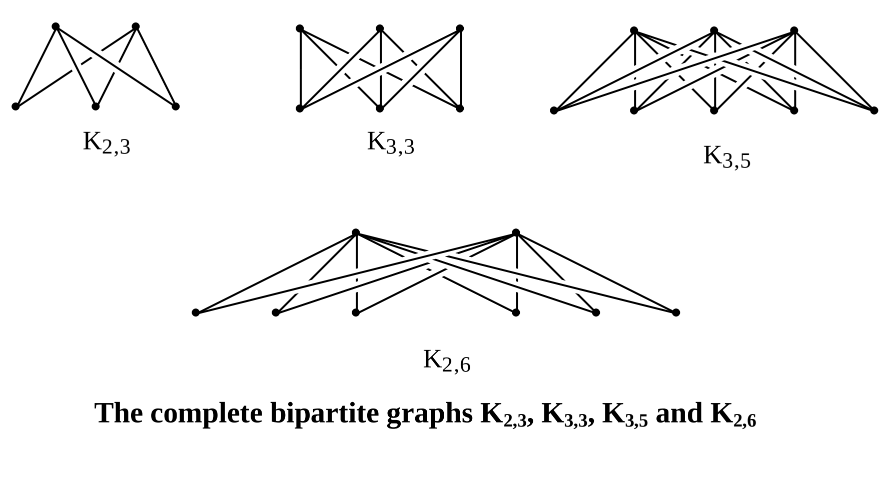
The simple graphs G_1, G_2 and G_3.

Solution: (i) G_1 is bipartite, since if we set V_1={v_1, v_2, v_3} and V_2={v_4, v_5} each edge is incident on one vertex in V_1 and one vertex in V_2.

(ii) G_2 is not bipartite. Let us consider V_1 and V_2 be two partitioned of V_G, and $v_5 \in V_1$. Then v_4 must be in V_2 and v_6 must be in V_1 by the edge e_2. Then v_5 must be in V_2 by the edge e_3. So $v_5 \in V_1$ and $v_5 \in V_2$.

(iii) G_3 is bipartite. The disjoint vertex sets are V_1={v_1, v_3, v_5} and V_2 = {v_2, v_4}.

Complete bipartite graph: The complete bipartite graph on m and n vertices, denoted by $K_{m,n}$ is the simple graph whose vertex set is partitioned into two sets V_1 with m vertices and V_2 with n vertices in which there is an edge between each pair of vertices {v_1, v_2} with $v_1 \in V_1$ and $v_2 \in V_2$. The complete bipartite graph $K_{2,3}$, $K_{3,3}$, $K_{3,5}$ and $K_{2,6}$ are displayed in fig-11.



The complete bipartite graphs $K_{2,3}$, $K_{3,3}$, $K_{3,5}$ and $K_{2,6}$

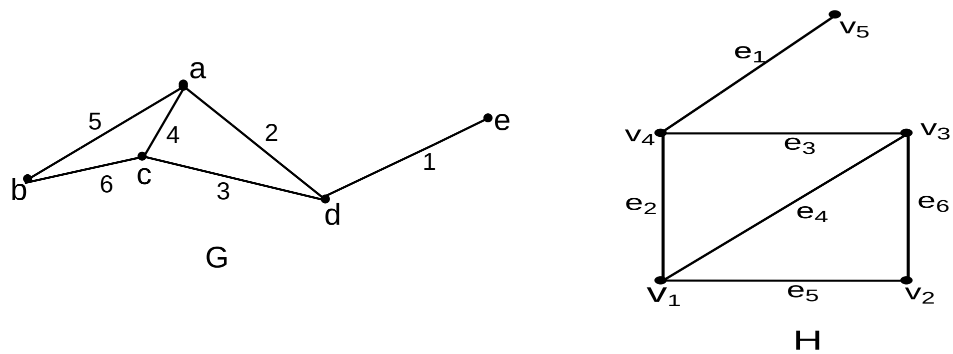
Graph Isomorphism and Subgraphs

Isomorphism of Graphs: It is important to understand what one means by two graphs being same or different. Two graphs may have different geometrical structures but still be the same graph according to our definition.

Definition : Two graphs G and H are isomorphic denoted by $G \cong H$ if there exists a one-to-one mapping f from V_G to V_H such that

$$uv \in E_G \Leftrightarrow \{f(u), f(v)\} \in E_H \text{ for all } u, v \in V_G \text{ in } G.$$

Hence G and H are isomorphic if the number of edges joining u to v in G is also the number of edges joining f(u) to f(v) in H.
In particular, isomorphism of two graphs preserves adjacency and non-adjacency between any two vertices.



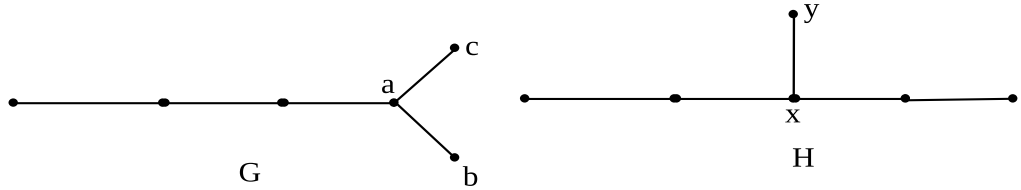
The simple graphs G and H.

The graphs G and H in Fig-13 are isomorphic. The vertices a, b, c, d and e correspond to v_1, v_2, v_3, v_4 and v_5 respectively. The edges 1, 2, 3, 4, 5 and 6 correspond to e_1, e_2, e_3, e_4, e_5 and e_6 respectively.

Observations: For any two isomorphic graphs must have the following properties:

1. The same number of vertices
2. The same number of edges
3. An equal number of vertices with a given degree.

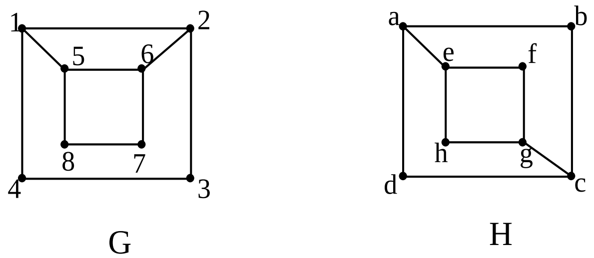
However, these conditions are not sufficient for an isomorphic graph. The following graphs G and H satisfy all the above conditions but yet they are not isomorphic.



The simple graphs G and H

The graph G in Fig-14 the vertex 'a' of degree 3 which is adjacent to two pendant vertices b and c and one vertex of degree 2 where as in H the vertex 'x' of degree 3 which is adjacent to only one pendant vertex y and two vertices of degree 2 each. Hence adjacency is not preserved. Therefore, G and H are not isomorphic.

Example1: Determine whether the following graphs are isomorphic.



The simple graphs G and H

Solution:

(i) Both the graphs G and H have 8 vertices and 10 edges.

(ii) Both have number of vertices of degree 2 are 4 and number of vertices of degree 3 are 4.

(iii) For adjacency, the vertex 1 in G of degree 3 is adjacent to one vertex of degree 3 and other vertex of degree 2, but in H there does not exist any vertex of degree 3 which is adjacent to two vertices of degree 3 and degree 2 respectively. Hence, adjacency is not preserved.

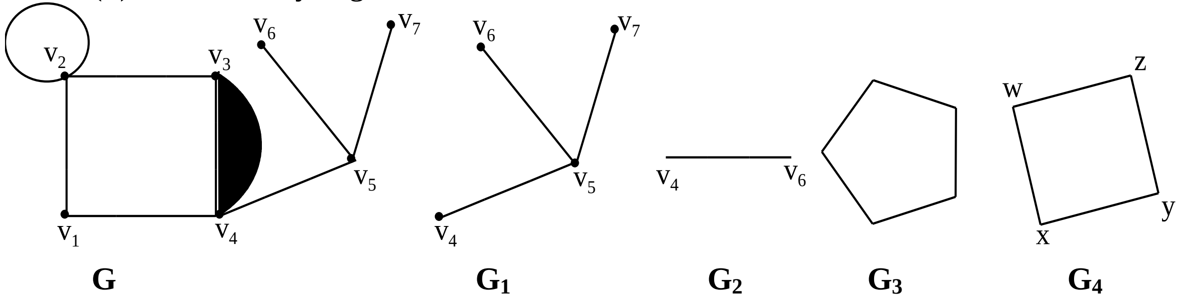
Therefore, G and H are not isomorphic.

Subgraphs: In some situation we deal with parts of the graph and a solution can be found to the problem by combining the information determined by parts. For example, the existence of an Euler tour (will see later on) is very local which depends only on the number of adjacent vertices. A graph H is said to be a subgraph of a graph G if it is obtained by selecting certain edges and vertices from G subject to the restriction that if we select an edge e in G that is incident on the vertices u and v, we must include u, v in H. The formal definition follows.

Definition 1. Let $G = (V_G, E_G)$ be a graph. $H = (V_H, E_H)$ be a subgraph of G if

(i) $V_H \subseteq V_G$ and $E_G \subseteq E_H$

(ii) For every edge $e' \in E_H$, if e' is incident on u' and v' then $u', v' \in V_H$.



The graphs G, G_1 G_2, G_3 and G_4

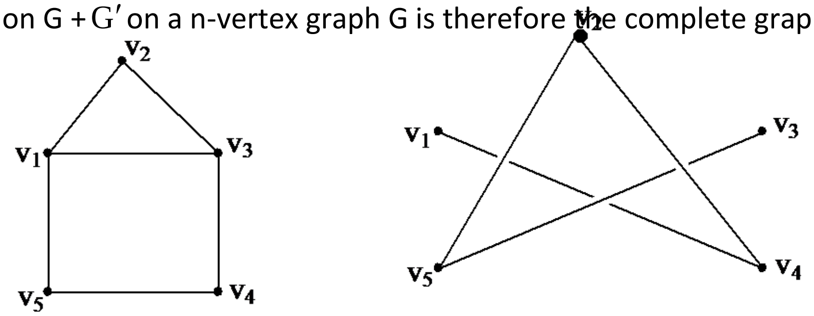
The graph G_1 in Fig-18. is a subgraph of G but G_2 is not a subgraph of G since there is no edge between v_4 and v_6 in G. G_3 is also not a subgraph of G because there is no pentagonal subgraph. G_4 is an isomorphic subgraph of G.

Observations

1. Every graph is its own subgraph
2. A subgrah of a subgraph of G is a subgraph of G
3. A single vertex in a graph G is a subgraph of G
4. A single edge together with its end vertices is also a subgraph of G.

<u>Complement of a graph:</u> Let G is a simple graph. The complement of G denoted by G' (sometimes denoted by $\overline{G}$) is the graph with the same vertex set V_G but edge set E_G contains of the edges not present in G (i.e. two vertices are adjacent in G if and only if they are not adjacent in G'). That is $E_{G'} = \{e \in E(V) : e \notin E_G\}$, where E(V) edge set of the complete graph K_V.

The graph union $G + G'$ on a n-vertex graph G is therefore the complete graph K_n.



G $\quad\quad\quad\quad$ G'

The simple graph G and its complement G'

Matrix Representation of Graphs

A geometrical representation of a graph has limited use. When a graph has many vertices and edges, it is essential to use a computer to perform graph algorithms. A matrix is convenient to represent a graph by means of numbers, which are easy to store and manipulate in computers than at recognizing pictures.

A graph is completely determined by specifying either its adjacency structure or its incidence structure. These specifications provide for more efficient ways of representing a large or complicated graph than a pictorial representation.

In this section we introduce the incidence, adjacency, circuit, path and cut-set matrices of a graph and establish several properties that help to reveal the structure of a graph. Not only these matrices are useful device for storing the basic structure of any graph, they manipulate in order to study its properties. The properties of these matrices and other related results to be established in this section will be used in other section of chapter-6 and chapter-7.

Adjacency Matrix

Suppose that a graph n vertices, numbered v_1, v_2, ….., $v_{n.}$. This numbering imposes arbitrarily on the set of vertices. Having ordered the vertices, we can form an $n \times n$ matrix where entry (i, j) is the number of edges between v_i and v_j.

Definition: Let $G(V_G, E_G)$ be a vertex-labeled simple graph of order n. The adjacency matrix $A = (a_{ij})_{n\times n}$ of G is the matrix in which the entry a_{ij} is defined as follows:

G is undirected:

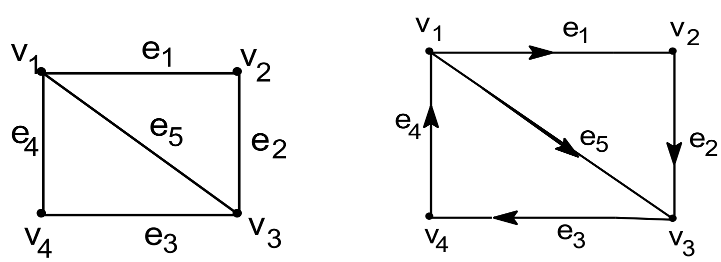
$$a_{ij} = \begin{cases} 1, & \text{when } v_i \text{ is adjacent to } v_j \\ 0, & \text{otherwise} \end{cases}$$

G is directed:

$$a_{ij} = \begin{cases} 1, & \text{if } (v_i, v_j) \in E \\ 0, & \text{otherwise} \end{cases}$$

where v_i and v_j are the vertices of G.

Example:



The simple graph G and the directed simple G(D)

The adjacency matrices for the graphs are

$$A_G = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix} \text{ and } A_{G(D)} = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix} \text{ respectively.}$$

Observations:

There are a number of observations that we can make about the adjacency matrix A of a graph G:

G is undirected simple graph:

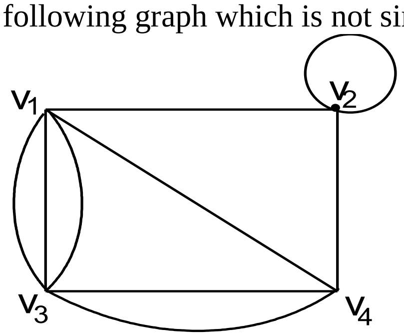
(i) A is symmetric.
(ii) A is a zero matrix if and only if G is a null graph.
(iii) The sum of entries in each row i of A equals to the degree sum of v_i..
(iv) Identity columns indicate the multiple edges.
(v) A is a 0-1 bit (or binary) matrix and there is a one-to-one correspondence between n vertices and n×n symmetric binary matrix with all entries on the leading diagonal equal to zero.
(vi) The (i, j) entry of A^m is the number of paths of length m from vertex v_i to the vertex v_j, in G.
(vii) If $i \neq j$ the (i, j) entry of A^2 is equal to the number of paths containing exactly two edges from v_i and v_j. The (i, i) entry of A^2 is the degree of v_i and that of A^3 is equal to twice the number of triangles containing v_i.
(viii) The (i, j) entry of $(A + A^2 + A^3 + \ldots. + A^n)$ gives the number of paths of length m or less from v_i to v_j.

G is directed simple graph:

(i) A is not necessarily symmetric
(ii) The sum of the entries in any column j of A is equal to the number of arcs directed towards v_j.
(iii) The sum of the entries in any row i is equal to the number of arcs directed away from vertex v_i.
(iv) The (i, j) entry of A^m is equal to the number of walks of length m from vertex v_i to v_j.
(v) The diagonal entries of $A \cdot A^T$ shows the out degree of the vertices of G and diagonal entries of $A^T \cdot A$ shows the in degree of the vertices of G.

Note: Adjacency matrices can also be used to represent multigraphs or pseudograph. These matrices are not a bit matrix.

Example: Consider the following graph which is not simple.



The graph G

The adjacency matrix for the above graph is $A_G = \begin{pmatrix} 0 & 1 & 3 & 1 \\ 1 & 2 & 0 & 1 \\ 3 & 0 & 0 & 2 \\ 1 & 1 & 2 & 0 \end{pmatrix}$.

Incidence Matrix:

Definition: Consider a graph G with n vertices and m edges having no self loops. The incidence matrix $B_G = (b_{ij})_{n\times m}$ of G has n rows, one for each vertex, and m columns one for each edge. The entry b_{ij} is defined as follows:

G is undirected:

$$b_{ij} = \begin{cases} 1, & \text{if the jth edge is incident on the ith vertex} \\ 0, & \text{otherwise} \end{cases}$$

G is directed:

$$b_{ij} = \begin{cases} 1 & \text{if the jth edge is incident on ith vertex and oriented away from it} \\ -1 & \text{if the jth edge is incident on ith vertex and oriented towards it} \\ 0, & \text{if the jth edge is not incident on the ith vertex.} \end{cases}$$

Example: Consider the graphs G and G(D) in Fig.-1. Find the incidence matrix.
The incidence matrices for the graphs are

$$B_G = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \begin{array}{c} \begin{array}{ccccc} e_1 & e_2 & e_3 & e_4 & e_5 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{pmatrix} \end{array} \text{ and } B_{G(D)} = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \begin{array}{c} \begin{array}{ccccc} e_1 & e_2 & e_3 & e_4 & e_5 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & -1 & 1 \\ -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & -1 \\ 0 & 0 & -1 & 1 & 0 \end{pmatrix} \end{array} \text{ respectively.}$$

Observations:

The following observations are made about the incidence matrix:

(i) Each entry is either 0 or 1.
(ii) Identical columns indicate the multiple edges.
(iii) A row with all zeros correspondence to an isolated vertex.
(iv) A row with single unit entry correspondence to a pendent vertex.
(v) The sum of entries along ith row is the degree of the corresponding vertex v_i.
(vi) If B is a 0-1 matrix, i.e. G is simple then there is a one-to-one correspondence between vertices and edges of B.
(vii) The graphs are isomorphic if and only if their corresponding incidence matrices differ only by a permutation of rows or columns.
(viii) If G is a connected graph with n vertices then the rank of B is n-1.

(ix)If G is disconnected and consist of two components G_1 and G_2, the incidence matrix B(G) can be written in block-diagonal form as

$$B(G) = \begin{bmatrix} B(G_1) & \vdots & 0 \\ \cdots\cdots & & \cdots\cdots \\ 0 & \vdots & B(G_2) \end{bmatrix}$$

Where $B(G_1)$ and $B(G_2)$ are the incidence matrices of components G_1 and G_2.

(x) If G has k components then rank of B is n-k.

Graph Connectivity

Many problems can be modeled with paths formed by travelling along the edges of the graph. For instance, the problem of determining whether a message can be sent between two computers using intermediate links can be studied with a graph model. Problems of efficiently planning routes for mail delivery, garbage pickup, and diagnostics in computer network and so on can be solved using models that involve paths in graphs. Some graph theoretical questions ask for optimal solutions to problems such as: find a shortest path (in a complex network) from a given point to another.

Definition: A walk or an edge sequence in a graph is defined as a finite alternating sequence of vertices and edges beginning and ending with vertices such that each edge in the sequence is incident with the vertices preceding it and succeeding it.

No edge appears more than once in a walk. A vertex however may appear more than once. The vertices at the beginning and at the end of a walk are called terminal vertices. Whenever the terminal vertices are same then the walk is called closed walk, otherwise open walk or trial.

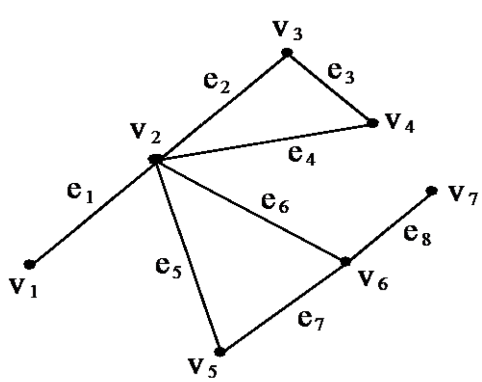
Definition: An open walk in which no vertex appears more than once is called a path.

A path of length n from v_0 to v_n is an alternating sequence of n +1 vertices and n edges beginning with v_0 and ending with v_n ($v_0e_1v_1e_2v_2\ldots..v_{n-1}e_nv_n$) in which edge e_i is incident on vertices v_{i-1} and v_i (i = 1, 2, …., n).

Definition: A circuit (cycle) is a closed path of non-zero length with no repeated edges.

A simple circuit is a cycle from v to v in which no vertex (except the starting vertex that appear twice) appears more that once. Clearly every vertex in a simple circuit is of degree two and does not contain the same edge more than once. Note that every self-loop is a circuit but every circuit is not a self-loop.

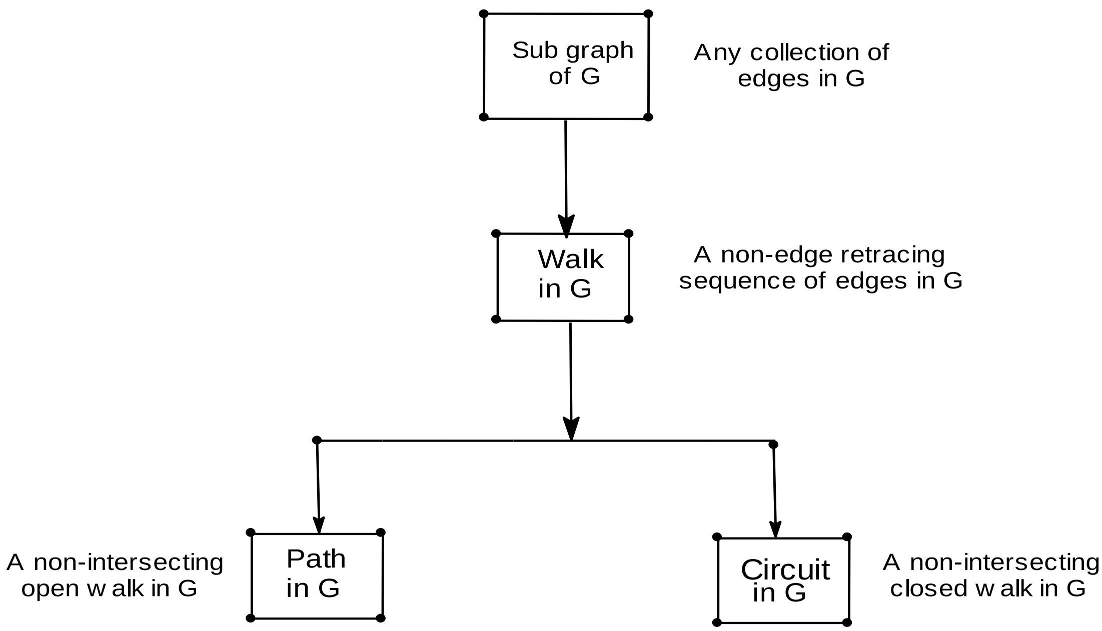
Example: For the graph G in Fig.- we have the following information



The graph G

Walk	Simple path	Cycle	Simple cycle
$(v_6e_7v_5e_5v_2e_4v_4e_3v_3e_2v_2e_1v_1)$	No	No	No
$(v_6e_7v_5e_5v_2e_4v_4)$	Yes	No	No
$(v_2e_6v_6e_7v_5e_5v_2e_4v_4e_3v_3e_2v_2)$	No	Yes	No
$(v_5e_7v_6e_6v_2e_5v_5)$	No	Yes	Yes
v_7	Yes	No	No

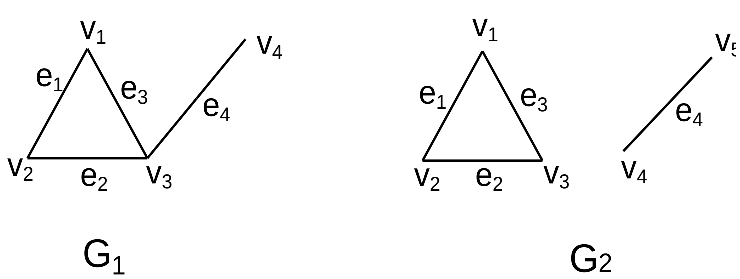
Observation: Following figure summarized the above definitions



Walk, paths and circuits as subgraphs

Connected Graphs and Components: When does a computer network have the property that every pair of components can share information, if message can be send through one or more intermediate computers, when a graph is used to represent this computer network, where vertices represent the computers and edges represents the communications links, this question become when is there always a path between two vertices in the graph.

Definition: A graph G is said to be a connected graph if there is a path between every pair of distinct vertices of G.



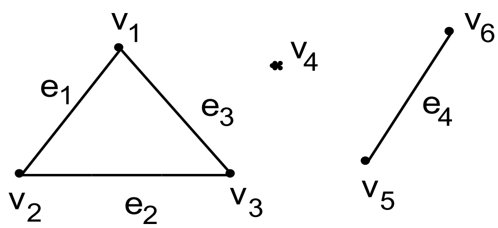
The graphs G_1 and G_2

Here G_1 is a connected graph, but G_2 is not a connected graph.

Components: A graph that is not connected is the union of two or more connected sub graphs, each pair of which has no vertices in common. These disjoint connected sub graphs are called the components of the graph.

Definition: Let G be a graph and v be a vertex in G. The sub graph G′ of G consisting of all edges and vertices of G that are contained in some path beginning at v is called the component of G.

Example: The component of the graph G in Fig. Containing v_3 is the subgraph $G_1 = (V_1, E_1)$ where $V_1 = \{v_1, v_2, v_3\}$. And $E = \{e_1, e_2, e_3\}$. Component of G containing v_4 is the subgraph $G_2 = (V_2, E_2)$ where $V_1 = \{v_4\}$ and $E_2 = \varphi$.



The graph G

Theorem 1: A graph is connected if and only if it has one and only one component.

Proof: Let G(V, E) be a connected graph. Then for any two vertices u, v $\in$ V there is a path from u to v. If possible let there exist two components containing the vertices v_i and v_j. Then by definition by component there does not exist any path between v_i and v_j. If we consider more that two components in the same way we arrive a contradiction. Hence there exists only one component.

Conversely, suppose that there exist one and only one component in G. Now we show that G is connected. If not there exist at least two vertices $v_i, v_j \in V$ for which there does not exist any path between v_i and v_j and we get two components one containing v_i and other containing v_j. This is a contradiction. Hence the graph is connected.

Theorem 2: If a graph (connected or disconnected) has exactly two vertices of odd degree, there must be a path joining these two vertices.

Proof: Let G be a connected graph with all vertices of even degrees except u and v, which are of odd degrees. We know that the number of odd degree vertices in a graph is always even. Therefore, for every component of a disconnected graph cannot have an odd number of odd degree vertices. Hence u, v must have in the same component of G and must have a path between them.

Theorem 3: In a simple graph with n vertices and m components can have at most $\frac{(n-m)(n-m+1)}{2}$ edges.

Proof: Let G be the simple graph with n vertices and m components. Let $K_1, K_2, \ldots, K_m$ be the components of G. Let n_i (i = 1, 2, …, m) be the number of vertices in K_i. Then $n_1 + n_2 + \ldots + n_m = n$.

Since the graph is simple maximum number of edges in K_i is $\frac{n_i(n_i-1)}{2}$.

So the maximum number of edges in G is $\sum_{i=1}^{m} \frac{n_i(n_i-1)}{2} = \frac{1}{2}\sum_{i=1}^{m} n_i^2 - \frac{n}{2}$.

Now $\sum_{i=1}^{m}(n_i-1) = n - m$.

Therefore, $\left\{\sum_{i=1}^{m}(n_i-1)\right\}^2 = (n-m)^2$

or $\sum_{i=1}^{m}(n_i-1)^2 + 2\sum_{i\neq j}(n_i-1)(n_j-1) = (n-m)^2$

or $\sum_{i=1}^{m} n_i^2 - 2n + m + 2\sum_{i\neq j}(n_i-1)(n_j-1) = (n-m)^2$

or $\sum_{i=1}^{m} n_i^2 \leq (n-m)^2 + 2n - m$.

Hence $\sum_{i=1}^{m}\frac{n_i(n_i-1)}{2} \leq \frac{1}{2}(n^2 + n^2 - 2nm + 2n - m) - \frac{n}{2} = \frac{(n-m+1)(n-m)}{2}$.

Shortest Path Problems

In this section, we are finding a shortest path between vertices in a weighted connected graph. The need to find shortest paths in graphs arises in many different situations. The path of least length between two vertices is the sum of the weights of the edges of the path. There are several algorithms to find the shortest path between two vertices in a weighted connected graph. We will present an algorithm by E. Dijkstra, one of the pioneers in computer science.

Another important problem involving weighted connected graphs asks for the circuits of minimal length (weight) that visit every vertex of a complete graph exactly once. This is the famous traveling salesman problem. We will discuss this problem later in this section.

Definition: Let G^w be an edge weighted graph, i.e. G^w is a graph $G(V_G, E_G)$ together with a weight function $w : E_G \to 3$ on its edges.

For any subgraph H of G, let $w(H) = \sum_{e \in E_G} w(e)$ be the total weight of H.

If $P = e_1e_2 \ldots e_n$ is a path, then its weight is $w(P) = \sum_{i=1}^{n} w(e_i)$. The shortest distance or minimum weighted distance between two vertices u and v is

$$d_G^w(u,v) = \min\left\{w(P) \mid P : u \to \ldots. \to v\right\}.$$

Dijkastra's Algorithm for shortest path in weighted graph

This algorithm finds the length of a shortest path from the vertex a to the vertex z in a weighted connected graph G = (V, E). G has the vertices $a = v_0, v_1, \ldots, v_n = z$ and weight of the edge $\{v_i, v_j\}$ is $w(v_i, v_j) > 0$ and the level of the vertex v_i is $L(v_i)$. $w(v_i, v_j) = \infty$ if $\{v_i, v_j\}$ is not an edge in G.

Input: A weighted connected simple graph G = (V, E) in which all weights are positive.

Output: L(z) the length of the shortest path from a to z.

Procedure Dejkstra

For i: = 1 to n

$L(v_i) : = \infty$

L(a) : = 0

$T : = \phi$

{T is the set of vertices whose shortest distance from a has not been found}

while $z \notin T$

begin

u : = a vertex not in T with minimum L(u)

$T : = T \cup \{u\}$

for all vertices not in T

if L(u) + w(u, v) < L(v) then L(v) : = L(u) + w(u, v)

{this adds a vertex to T with minimum lable and updates the labels of vertices not in T}

end { L(z) = length of the shortest path from a to z}.

end Dijkstra.

Example 1: Apply Dijkstra's algorithm to the graph given below and find the shortest path from s to z.

Solution: The initial labeling is given by.

Vertex V	s	a	b	c	d	z
$\lambda(v)$	0	∞	∞	∞	∞	∞
T	{s,	a,	b,	c,	d,	z}

Iteration 1: u = s has $\lambda(u) = 0$. There are two edges incident with s i.e. {s, a} and {s, b}. Both a, b in T, $\lambda(a) = \infty > 2 = 0 + 2 = \lambda(s) + w\{s, a\}$. So $\lambda(a)$ becomes 2, and similarly $\lambda(b)$ becomes 3. Now T becomes T - {s} i.e. s is coloured. Thus

Vertex V	s	a	b	c	d	z
$\lambda(v)$	0	2	3	∞	∞	∞
T	{	a,	b,	c,	D,	z}

Iteration 2: Again, u = a has $\lambda(u)$ a minimum and $u \in T$. There are two edges incident on a i.e. {a, b} and {a, d}. Since $\lambda(d) = \infty > 4 = 2 + 2 = \lambda(a) + w\{a, d\}$. So $\lambda(d)$ becomes 4, and $\lambda(b) = 3 < 4 = 2 + 2 = \lambda(a) + w\{a, b\}$so $\lambda(b)$ becomes 3.

Now T becomes T - {a} i.e. s is colored. Thus

Vertex V	s	a	b	c	d	z
$\lambda(v)$	0	2	3	∞	4	∞
T	{		b,	c,	d,	z}

Iteration 3: Again, u = b has $\lambda(u)$ a minimum for u $\in$T. There are two edges incident on b i.e. {b, c} and {b, d}. Since $\lambda(d) = 4 < 8 = 3 + 5 = \lambda(b) + w\{b, d\}$. So $\lambda(d)$ becomes 4, and $\lambda(c) = \infty > 5 = 3 + 1.5 = \lambda(b) + w\{b, c\}$so $\lambda(b)$ becomes 5.
Now T becomes T - {b} i.e. s is colored. Thus

Vertex V	s	A	b	c	d	z
$\lambda(v)$	0	2	3	5	4	∞
T	{			c,	d,	z}

Iteration 4: Again, u = d has $\lambda(u)$ a minimum for u $\in$T. There are two edges incident on d i.e. {d, c} and {d, z}. Since $\lambda(c) = 5 < 7 = 4 + 3 = \lambda(d) + w\{d, c\}$. So $\lambda(c)$ remains as it is and $\lambda(z) = \infty > 8 = 4 + 4 = \lambda(d) + w\{d, z\}$so $\lambda(c)$ becomes 7.
Now T becomes T - {d} i.e. s is colored. Thus

Vertex V	s	A	b	c	d	z
$\lambda(v)$	0	2	3	5	4	8
T	{			c,		z}

Iteration 5: u = c has $\lambda(u)$ a minimum for u $\in$T. There is one edge incident on c i.e. {c, z}. Since $\lambda(z) = 8 > 7 = 5 + 2 = \lambda(c) + w\{d, z\}$. So $\lambda(f)$ becomes 7.
Now T becomes T - {c} i.e. s is colored. Thus

Vertex V	s	A	b	c	d	z
$\lambda(v)$	0	2	3	5	4	7
T	{					z}

u = z, the only one choice
hence stop.

Thus the length of shortest paths from s to a, b, c, d and z are 2, 3, 5, 4 and 7 respectively.

Euler and Hamiltonian Paths

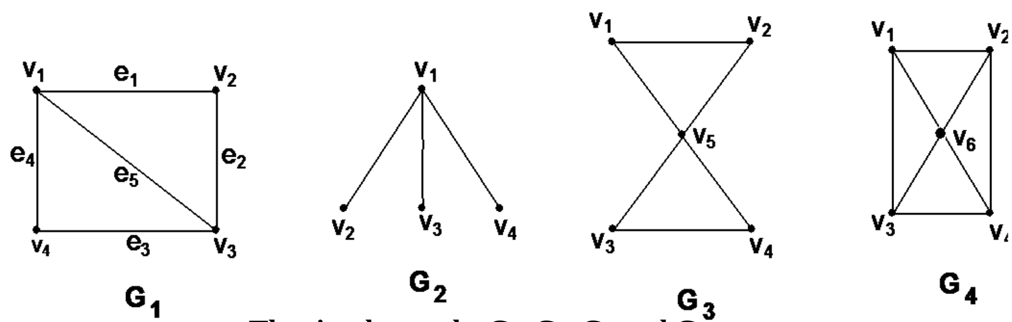
Euler trial and Tour:

Definition: An Euler trial in a graph G is an open walk containing every edges of G. An Euler tour in a graph G is a closed walk containing every edges of G. A graph that has an Euler tour is called an Euler graph.

Note: (i) If a graph consists of only one vertex v and no edges, then the path (v) is an Euler tour of G.

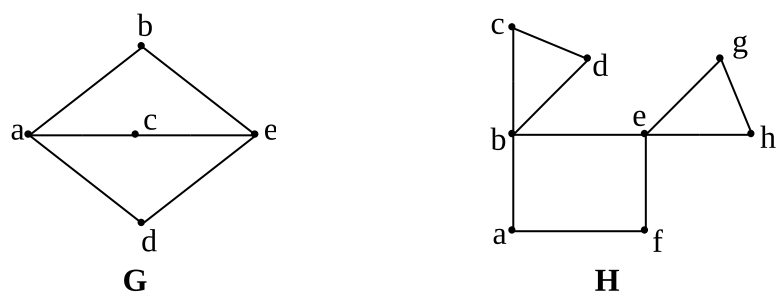
(ii) A graph that has an Euler trial may not have an Euler tour.

Example 1: The graph G_1 in Fig- has an Euler trial $v_1e_1v_2e_2v_3e_3v_4e_4v_1e_5v_3$ or (v_1, v_2, v_3, v_4, v_1, v_3) but in G_2 does not have an Euler trial.
Similarly, in G_3 has an Euler tour (v_1, v_5, v_3, v_4, v_5, v_2, v_1) but in G_4 does not have an Euler tour.



The simple graphs G_1, G_2, G_3 and G_4

Example 2: Determine whether Eulerian trial and tour exist in the following graphs in Fig-.



Solution: Since Eulerian trial and tour traverses each edge in a graph once and only once in G Eulerian trial is (a, c, e, b, a, d, e) and there is no Eulerian tour. In graph H Eulerian trial is (a, f, e, h, g, e, b, d, c, b, a) and the Eulerian tour is also the same.

Improper Integral

INTRODUCTION: When dealing with different problems of science and technology we have to face different integrations where either the limits a and b are finite or the integrand *f(x)* is unbounded in $a \leq x \leq b$. These type of integrals are called improper integral.

Definition of Improper Integral: A definite integral $\int_a^b f(x)dx$ is called improper integral if either,

i) A limit is infinite or both are infinite i.e., $a = \infty$ or $b = \infty$; or both.

ii) The integrand becomes infinite in $a \leq x \leq b$.

Example: Some improper integrals are $\int_0^\infty \frac{1}{x+1} dx$, $\int_1^3 \frac{1}{(x-1)(x-3)} dx$, $\int_{-\infty}^\infty \frac{1}{x+1} dx$, etc.

Example solved: Verify the improper integral $\int_0^1 \frac{1}{x} dx$ exists or not.

Solution: Here f*(x)*=$\frac{1}{x}$, which has an infinite discontinuity at the left end point *x=0.*

Therefore $\int_0^1 \frac{1}{x} dx = = \lim_{\varepsilon \to 0+} \int_{0+\varepsilon}^1 \ \frac{1}{x} dx$

$= \lim_{\varepsilon \to 0+} [\log \ x\,]_\varepsilon^0$

$= \lim_{\varepsilon \to 0+} \{\log 1 - \log \varepsilon\}$

$= -\infty$.

Therefore the given improper integral does not exist.

Gamma function: The improper integral $\Gamma(n) = \int_0^\infty e^{-x} x^{n-1} dx$ for $n > 0$,

is called Gamma Function.

Properties of Gamma function:

1. For $a > 0$, $\int_0^\infty e^{-ax} x^{n-1} dx = \frac{\Gamma(\mathrm{n})}{\mathrm{a}^\mathrm{n}}$, for $n > 0$.
2. $\Gamma(\mathrm{n}+1) = \mathrm{n}\Gamma(\mathrm{n})$, $n > 0$.
3. $\Gamma(1) = 1$.
4. When n is positive integer, $\Gamma(\mathrm{n}+1) = \mathrm{n}!$.
5. $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$.

Beta function: The improper integral $B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx$ for $m,n > 0$,

is called Beta Function.

Properties of Beta function:

1. $B(m,n) = B(n,m)$ for $m,n > 0$.
2. $B(m,n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}dx = \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}dx$, for $m,n > 0$.
3. $B(m,n) = 2\int_0^{\frac{\pi}{2}} sin^{2m-1}x\, cos^{2n-1}x\, dx$ for $m,n > 0$.
4. $B(m,n) = \frac{\Gamma(\mathrm{m})\Gamma(\mathrm{n})}{\Gamma(\mathrm{m+n})}$, for $m,n > 0$.

Example: $\int_0^{\frac{\pi}{2}} sin^6x\, cos^5x\, dx$.

Here $\int_0^{\frac{\pi}{2}} sin^6x\, cos^5x\, dx = \frac{1}{2}\frac{\Gamma\left(\frac{7}{2}\right)\Gamma(3)}{\Gamma(\frac{7}{2}+3)} = \frac{1}{2}\frac{\Gamma\left(\frac{7}{2}\right)\Gamma(3)}{\frac{11}{2}\cdot\frac{9}{2}\cdot\frac{7}{2}\Gamma(\frac{7}{2})} = \frac{1}{2}\frac{\Gamma(3)}{\frac{11}{2}\cdot\frac{9}{2}\cdot\frac{7}{2}} = \frac{1}{2}\frac{2!}{\frac{11}{2}\cdot\frac{9}{2}\cdot\frac{7}{2}} = \frac{8}{693}$.

Laplace Transforms

Definition of Laplace Transform: Let $y = F(t)$, the Laplace transform of *y* is defined by $L\{F(t)\} = \int_0^\infty e^{-st}F(t)dt$.

The transformed function is called $f(s)$, thus $L\{F(t)\} = f(s)$.

Formulae of Laplace Transform:

1. $\mathrm{L}(1) = \frac{1}{s}$
2. $L(t^n) = n!/(s^{n+1})$
3. $L(t) = \frac{1}{s^2}$
4. $L(e^{at}) = 1/(s-a)$
5. $L(sinat) = a/(s^2+a^2)$
6. $L(cosat) = s/(s^2+a^2)$
7. $L(sinhat) = a/(s^2-a^2)$
8. $\mathrm{L}\,(coshat) = s/(s^2-a^2)$

Existence of Laplace transform:

If the function F(t) satisfies the following two conditions ,then its Laplace transform exists;

(i) Every interval [0,N] can be subdivided into a finite number of intervals in each of which the function is continuous and has finite right and left limits.

(ii) There exist a real constant M>0 and α such that for all t>N

$$|f(t)| \le Me^{\alpha t}, \quad \forall t \ge 0.$$

Properties of Laplace Transform:

Property 1. Linearity Property

If *a* and *b* are constants while *F*(*t*) and *G*(*t*) are functions of *t*, then

$$\boldsymbol{L\{a \cdot F(t) + b \cdot G(t)\} = a \cdot L\{F(t)\} + b \cdot L\{G(t)\}}$$

Example 1: $L\{3t + 6t^2\} = 3 \cdot L\{t\} + 6 \cdot L\{t^2\}$

<u>Property 2. Change of Scale Property</u>

If $L\{F(t)\} = f(s)$ then $\boldsymbol{L\{F(at)\} = (1/a)f(s/a)}$

Example 2: $L\{F(5t)\} = (1/5)f(s/5)$.

<u>Property 3. First Shifting Property</u>

Let $L\{F(t)\} = f(s)$ with a constant, then $\boldsymbol{L\{e^{at}F(t)\} = f(s-a)}$.

Example 3: Find L$\{e^{-5t}\ sin2t\}$.

Solution: We know $L\{\ sin2t\} = \frac{2}{s^2+2^2} = f(s)$(say). Therefore, by first shifting property, L$\{e^{-5t}\ sin2t\}$=$f\big(s-(-5)\big)$=$f(s+5)$=$\frac{2}{s^2+10s+29}$.

Second Shifting Property :

If $L\{F(t)\} = f(s)$ with a constant, and $G(t) = F(t-a), \quad t > a$

$= 0, \quad , \quad t < a$.

Then $\boldsymbol{L\{G(t)\} = e^{-as}f(s)}$.

Example 4: Find L$\{F(t)\}$ where F(t) is defined as,

$$F(t) = cos\left(t - \frac{2\pi}{3}\right), \qquad t > \frac{2\pi}{3}$$

$$=0, \qquad t < \frac{2\pi}{3}.$$

Solution: We know L$\{\ cos\ t\}$=$\frac{s}{s^2+1} = f(s)$

Therefore , by second shifting property, L{F(t)}= $e^{\frac{-2\pi s}{3}}f(s)$=$e^{\frac{-2\pi s}{3}}\frac{s}{s^2+1}$.

Property 4. <u>Laplace Transform of the Derivatives</u>

<u>First Derivative:</u>

If $L\{F(t)\} = f(s)$ then $\boldsymbol{L\{F'(t)\} = sf(s) - F'(0)}$.

<u>Second Derivative:</u>

. If $L\{F(t)\} = f(s)$ then $\boldsymbol{L\{F''(t)\} = s^2f(s) - sF'(0) - F''(0)}$.

n-th Derivative:

.If $L\{F(t)\} = f(s)$ then

$\boldsymbol{L\{F^n(t)\} = s^n f(s) - s^{n-1}F'(0) - s^{n-2}F''(0) - \cdots sF^{n-1}(0) - F^n(0)}$.

Laplace Transform on Integrals:

If $L\{F(t)\} = f(s)$, then $\boldsymbol{L\{\int_0^t F(x)dx\} = \frac{1}{s}f(s)}$.

Example 5: Find L$\{\int_0^t e^t cos2tdt\}$**.**

Solution: Let L$\{(cos2t)$=f(s). Then L$\{\int_0^t e^t cos2tdt\}$=$\frac{1}{s}f(s) = \frac{1}{s}L(e^t cos2t) =$ $\frac{1}{s}\varphi(s-1), by\ shifting\ property, where\ \varphi(s) = L(cos2t) = \frac{s}{s^2+4}. = \frac{1}{s}s - \frac{s-1}{(s-1)^2+4}$ $= s - 1/s(s^2$-2s+5).

Multiplication by $\boldsymbol{t^n}$:

If $L\{F(t)\} = f(s)$, then $\boldsymbol{L\{t^n F(t)\} = (-1)^n \frac{d^n}{ds^n} f(s)}$**.**

Example 6: Find L$\{t^2 cos5t\}$.

Solution: We know, L$(cos5t)$=$\frac{s}{s^2+5^2}$.

Therefore, L$\{t^2 cos5t\} = (-1)^2 \frac{d^2}{ds^2}\left(\frac{s}{s^2+25}\right) = 2s^3 - 150s/(s^2+25)^3$.

Division by t:

Let both $F(t)$and $\frac{F(t)}{t}$ have Laplace transforms and let $f(s)$ denote the Laplace transform of $F(t)$. If $\lim_{t\to 0}\frac{F(t)}{t}$ exists then $L\left\{\frac{F(t)}{t}\right\} = \int_s^\infty f(s)ds$.

Example 7: Determine the Laplace Transform of $\frac{\sin^2 t}{t}$.

Solution: The Laplace Transform of $\sin^2 t$ can be evaluated by

$$L\{\sin^2 t\} = L\{\frac{1 - \cos 2t}{2}\} = \frac{1}{2s} - \frac{1}{2}\frac{s}{s^2+4} = \frac{2}{s(s^2+4)}$$

$$\text{Thus,}\quad L\{\frac{\sin^2 t}{t}\} = \int_s^\infty \frac{2}{s(s^2+4)}\,ds = \frac{1}{4}\ln\frac{s^2+4}{s^2}\ .$$

Laplace Transform of Periodic Function:

Let F(t) be a periodic function of period T(>0), then

$$L\{F(t)\} = \frac{1}{1-e^{-sT}}\int_0^T e^{-sT}F(t)dt$$

Laplace Transform of Unit Step Function:

Definition of unit step function:

Let $a > 0$. Then, the unit step function $U_a(t)$ is

$$U_a(t) = 0, for\ t < a$$
$$= 1, for\ t > a$$

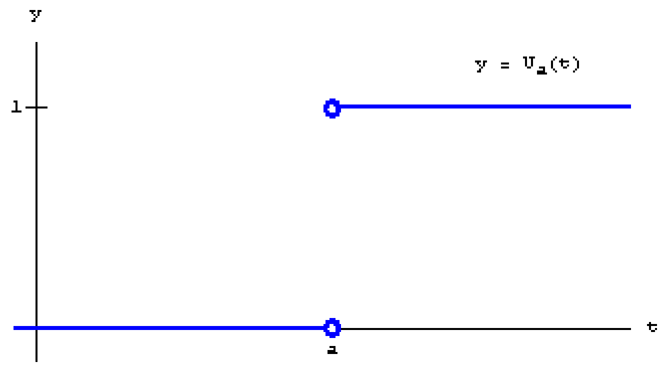


Figure 1 The graph of the unit step function $y = U_a(t)$.

Theorem 1: The function $F(t) = \ F_1(t)\ ,\ \ t < a$

$= \ F_2(t)\ ,\ \ t > a$ can be expressed by a unit step function like $F(t) = \ F_1(t)\ + \{\ F_2(t) - F_1(t)\}\, u(t-a)$.

Theorem 2: If

$$F(t) = \ F1(t)\ ,\ \ t < a_1$$
$$= \ F2(t)\ ,\ \ a_1 < t < a_2$$
$$= \ F3(t)\ ,\ \ a_2 \ < t,$$

then, $F(t)$ can be expressed as

$$F(t) = \ F_1(t)\ + \{F_1(t) - \ F_1(t)\}\, u(t - \ a_1)\ + \{F_3(t) - \ F_2(t)\}\, u(t - \ a_2).$$

Theorem 3: If $u(t-a)$ is a unit step function, then $L\{u(t-a)\} = \frac{e^{-as}}{s}$.

Theorem 4: If $L\{F(t)\} = f(s)$ and $u(t-a)$ be a unit step function, then

$$L\{F(t-a)u(t-a)\} = e^{-as}f(s).$$

Corollary: L{F(t)u(t-a)}=$e^{-as}L\{F(t+a)\}$.

Example 7: a) Find L{F(t)} whereF(t) $= e^{-t}, \quad 0 < t < 3$

$= 0, \quad t > 3.$

b) Find L{F(t)} where F(t) $= cost, \quad 0 < t < \pi$

$= cos2t\ , \quad \pi < t < 2\pi$

$= cos3t, \quad 2\pi < t.$

INVERSE LAPLACE TRANSFORM

Definition:

If $L\{F(t)\} = f(s)$, then $F(t)$ is called the Inverse Laplace Transform of $f(s)$ and is written as $L^{-1}\{f(s)\} = F(t)$.

Formulae of Inverse laplace Transform:

$\boldsymbol{F(t)}$	$\boldsymbol{f(s)}$
$\frac{1}{s}$	1
$\frac{1}{s^2}$	t
$\frac{n!}{s^{n+1}}$	t^n
$\frac{1}{s-a}$	e^{at}
$\frac{a}{s^2+a^2}$	$sinat$
$\frac{s}{s^2+a^2}$	$cosat$
$\frac{a}{s^2+a^2}$	$sinhat$
$\frac{s}{s^2+a^2}$	$coshat$

<u>Linear Property of Inverse Laplace Transform</u>:

If $L\{F_1(t)\} = f_1(s)$ and $L\{F_2(t)\} = f_2(s)$,then

$L^{-1}\{c_1 f_1(s) + c_2 f_2(s)\} = c_1 L^{-1}\{f_1(s)\} + c_2 L^{-1}\{f_2(s)\}$.

PROBLEM1: Find $L^{-1}\left\{\frac{4}{s-2} - \frac{3s}{s^2+16} + \frac{5}{s^2+4}\right\}$.

Shifting Property of Inverse Laplace Transform:

Theorem1: (First Shifting Property)

$L^{-1}\{f(s-a)\} = e^{at}L^{-1}\{f(s)\}$.

PROBLEM1: Find $L^{-1}\{\frac{4s+12}{s^2+8s+16}\}$.

Theorem2: (Second Shifting Property)

$If\ L^{-1}\{f(s)\} = F(t), then\ L^{-1}\{e^{-as}f(s)\} = F(t-a), \quad t > a$

$0, \quad t < a.$

PROBLEM2: Find $L^{-1}\{\frac{8e^{-3s}}{s^2+4}\}$.

Change of Scale Property of Inverse Laplace Transform:

$If\ L^{-1}\{f(s)\} = F(t), then\ L^{-1}\{f(as)\} = \frac{1}{a}F\left(\frac{t}{a}\right),$ a is a constant.

PROBLEM3 : Find $L^{-1}\left\{\frac{3s}{4s^2+16}\right\}$.

Inverse Laplace Transform on Derivatives:

Theorem1:(on 1st order derivatve)

If $L^{-1}\{f'(s)\} = -tF(t)$.

PROBLEM 4: Find $L^{-1}\left\{\frac{s}{(s^2+9)^2}\right\}$.

Theorem2:(on n^{th} order derivatve)

$If\ L^{-1}\{f(s)\} = F(t), then\ L^{-1}\{f^n(s)\} = (-1)^n t^n F(t).$

PROBLEM5: Find $L^{-1}\left\{log\,\frac{s+3}{s+2}\right\}$.

Multiplication by s^n:

Theorem1: $If\ L^{-1}\{f(s)\} = F(t) and\ F(0), then$

$L^{-1}\{sf(s)\} = F'(t).$

Theorem2: $If\ L^{-1}\{f(s)\} = F(t) and\ F(0) = F'(0) = .. =$ $F^{n-1}(0) = 0, then\ L^{-1}\{s^n f(s)\} = F^n(t).$

PROBLEM6: Find $L^{-1}\left\{\frac{s}{(s+1)^5}\right\}$.

Division by s: $If\ L^{-1}\{f(s)\} = F(t), then\ \boldsymbol{L^{-1}\{\frac{f(s)}{s}\} = \int_0^t F(u)du.}$

PROBLEM 7: Find $L^{-1}\left\{\frac{1}{s^2(s^2+1)}\right\}$.

Inverse Laplace Transform of Integrals:

$$If\ L^{-1}\{f(s)\} = F(t), then$$

$$L^{-1}\{\int_s^{\infty} f(u)du\} = \frac{F(t)}{t}.$$

PROBLEM 8: Find $L^{-1}\left\{\frac{s}{(s^2+9)^2}\right\}$.

Convolution property of Inverse Laplace Transform:

Definition of convolution of two functions: (Convolution). Let f and g be piecewise continuous functions for t ≥ 0. Then the **convolution** of f and g denoted by f∗g, is defined by the integral

$(f * g)(t) = \int_0^t f(u)g(t-u)du$.

Convolution theorem; Let f and g be piecewise continuous and of exponential order for t ≥ 0, then the Laplace transform of $f * g$ is given by the product of the Laplace transform of f and the Laplace transform of g. That is

$$\boldsymbol{L\{f * g\} = F(s)\,G(s)}.$$

PROBLEM 9: Using Convolution find (a) $L^{-1}\{s/(s^2+1)^2\}$.

Method of partial fraction to find Laplace Inverse Transform : Laplace Inverse Transform can also be evaluated by using method of partial fraction.

PROBLEM 10: Find $L^{-1}\{\frac{2s^2-4}{(s-3)(s^2-s-2)}\}$.

PROBLEM 10: Find y if $L\{y\} = \frac{12}{s^2-2s-8}$.

Solution:

$$s^2 - 2s - 8 = (s-4)(s+2)$$

$$\frac{12}{s^2-2s-8} = \frac{A}{s-4} + \frac{B}{s+2}$$

Solving for A and B, we get $A = 2$ and $B = -2$.

Therefore, $\frac{12}{s^2-2s-8} = \frac{2}{s-4} - \frac{2}{s+2}$

$$y(t) = L^{-1}\left\{\frac{12}{s^2-2s-8}\right\} = 2(e^{4t} - e^{-2t})$$

.

SOLUTION OF ODE USING LAPLACE TRANSFORM:

A linear differential equation with constant coefficient can be solved with the help of Laplace Transform. Given a differential equation of y(t) ,we apply Laplace Transform on both side .Applying necessary property of Laplace Transform, we find L{y(t)} as function of a variable, say $\varphi(s)$. Then y(t)}=$L^{-1}\{(\varphi(s)\}$. $This\ L^{-1}\{(\varphi(s)\}$ is obtained by applying several theorems and properties of inverse Laplace Transform.

Problem1: Solve , by Laplace transform, the differential equation

$y'' - 4\,y = 0, \quad y(0) = 1,\ y'(0) = 2$.

[Solution]

$$L\{\,y'' - 4\,y\,\} = L\{\,0\,\}$$

$or \qquad L\{\,y''\} - 4\,L\{\,y\,\} = 0$

$or, \qquad s^2\,L\{\,y\,\} - s\,y(0) - y'(0) - 4\,L\{\,y\,\} = 0$

$or \qquad s^2 L\{\,y\,\} - s - 2 - 4\,L\{\,y\,\} = 0$

$or, \qquad L\{\,y\,\} = \frac{s+2}{s^2+4}$